Midterm – Solutions

Problem M.1: (10 points)

(a) We have a difference equation

$$y[n] = \sqrt{2}y[n-1] - y[n-2] + x[n].$$

We can obtain the system function

$$Y = \sqrt{2}DY - D^2Y + X \qquad \frac{Y}{X} = \frac{1}{1 - \sqrt{2}D + D^2}.$$

The block diagram may be readily drawn. A DF II may be used.

And finally, the impulse response may be obtained by performing a partial fraction expansion:

- Replacing D by z^{-1}: $1 - \sqrt{2}z^{-1} + z^{-2}$
- Factoring the lowest power of z: $z^2 - \sqrt{2}z + 1$
- Finding the roots: $\Delta = 2 - 4 = -2 = (\sqrt{2}j)^2$. The roots are

$$p_1 = \frac{1+j}{\sqrt{2}} = e^{j\pi/4} \qquad p_1 = \frac{1-j}{\sqrt{2}} = e^{-j\pi/4}$$

- In conclusion: $1 - \sqrt{2}D + D^2 = (1 - e^{j\pi/4}D)(1 - e^{-j\pi/4}D)$

Performing the partial fraction expansion:

$$\frac{Y}{X} = \frac{1}{(1 - e^{j\pi/4}D)(1 - e^{-j\pi/4}D)} = \frac{1/(1+j)}{(1 - e^{j\pi/4}D)} + \frac{1/(1-j)}{(1 - e^{-j\pi/4}D)}.$$

Finally, the impulse response is

$$\begin{aligned} h[n] &= \left[\frac{1}{1+j}e^{j\pi n/4} + \frac{1}{1-j}e^{-j\pi n/4}\right] u[n] \\ &= \left[\frac{1}{\sqrt{2}}e^{-j\pi/4}e^{j\pi n/4} + \frac{1}{\sqrt{2}}e^{j\pi/4}e^{-j\pi n/4}\right] u[n] \\ &= \sqrt{2}\cos[\pi(n-1)/4]\, u[n] \end{aligned}$$

(b) The system is causal as it is apparent from the impulse response that is a causal function.

(c) The system is not stable. Indeed, the impulse response does not go to zero. Alternatively there are two poles on the unit circle.

Problem M.2: (10 points)

Examining the denominator of the rational function, we identify the roots:

$$z^3 - 5z^2 + 8z - 4 = (z-1)(z^2 - 4z + 4) = (z-1)(z-2)^2.$$

Checking the numerator,

$$z^4 - 1 = (z^2-1)(z^2+1) = (z-1)(z+1)(z^2+1),$$

and hence 1 is also a zero. Therefore, we have a pole-zero cancellation, and 1 is not really a pole. We have only a double pole at 2. Performing a partial fraction expansion yields:

$$\begin{aligned} X(z) &= \frac{z^4-1}{z^3-5z^2+8z-4} = \frac{z^4-1}{(z-1)(z-2)^2} = \frac{(z+1)(z^2+1)}{(z-2)^2} \\ &= z + 5 + \frac{17z-19}{(z-2)^2} \\ &= z + 5 + \frac{15}{(z-2)^2} + \frac{17}{z-2} \end{aligned}$$

The possible ROC's are rings delimited by poles and hence we have two choices

(i) $|z| < 2$ (0 included)

The inverse transform is here

$$\begin{aligned} x[n] &= \delta[n+1] + 5\delta[n] + \frac{15}{4}(-n+1)\frac{1}{2^{-n}}u[-n] - \frac{17}{2}\frac{1}{2^{-n}}u[-n] \\ &= \delta[n+1] + 5\delta[n] - \frac{1}{4}(15n+19)2^n u[-n] \\ &= \frac{1}{4}\delta[n] + \frac{1}{2}\delta[n+1] - (15n+19)2^{n-2}u[-n-2]. \end{aligned}$$

The signal is anti-causal because the ROC extends inward and 0 is included. If it were the impulse response of a DT LTI system, that system will be stable because the ROC includes the UC.

(ii) $2 < |z|$ ("infinity" not included)

The inverse transform is here

$$\begin{aligned} x[n] &= \delta[n+1] + 5\delta[n] + 15(n-1)2^{n-2}u[n-2] + \frac{17}{2}2^n u[n] \\ &= \delta[n+1] + \frac{27}{2}\delta[n] + 17\delta[n-1] + \frac{1}{4}(15n+19)2^n u[n-2]. \end{aligned}$$

The signal is right sided but not causal because the ROC extends outward and "infinity" is not included. If it were the impulse response of a DT LTI system, that system will not be stable because the ROC does not include the UC.

Problem M.3: (15 points)

(a) Since the cosine function is

$$\cos(z) = \frac{1}{2}\left[e^{jz} + e^{-jz}\right],$$

we can see that it does not have any point of singularity on the complex domain. However it is not defined at "infinity" and therefore the only possible ROC is $\mathbb{C}$.

(b) Consider a trigonometric contour around "0". Denoting by a_l the singularities of $H(z)z^{n-1}$ inside the contour,

$$h[n] = \sum \mathsf{Res}(\cos(z)z^{n-1}, a_l).$$

If $n \geq 1$ there are no singularities and hence $h[n] = 0$. For $n < 1$, the function $\cos(z)z^{n-1}$ has a pole at "0" of order $m = 1 - n$. Therefore

$$h[n] = \frac{1}{(-n)!}\left.\frac{d^{(-n)}\cos(z)}{dz^{(-n)}}\right|_{z=0} = \begin{cases} 0 & (-n) \text{ odd} \\ \frac{(-1)^{-n/2}}{(-n)!} & (-n) \text{ even} \end{cases}$$

(c) The impulse response could have been obtained by expanding the cosine:

$$\cos(z) = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots$$

Afterward, one may identify the impulse response by tracking the coefficients in the Laurent series.

(d) If $H(z)$ were the system function of a DT LTI system, the system *will not be* causal and actually, it will be anti-causal. This may be deduced from the time-domain function or from the fact the ROC does not contain infinity but 0.

The system will be stable because in the time domain we have convergence to zero and in the Z-domain the UC is inside the ROC.

Problem M.4: (10 points)

Since the Z-transform is linear, we can compute the transformation by looking at individual terms. The first term is a delta and hence the Z-transform is the constant "1".

Looking at the second term denoted $v[n]$, we note that

$$\begin{aligned} nv[n] = (1/\omega_o)\sin(\omega_o n)u[n-1] &= \frac{1}{2\omega_o j}\left[e^{j\omega_o n} - e^{-j\omega_o n}\right]u[n-1] \\ &= \frac{1}{2\omega_o j}\left[e^{j\omega_o n} - e^{-j\omega_o n}\right]u[n] \\ &= \frac{1}{2\omega_o j}\left(e^{j\omega_o}\right)^n u[n] - \frac{1}{2\omega_o j}\left(e^{-j\omega_o}\right)^n u[n]. \end{aligned}$$

Therefore,

$$
\begin{aligned}
-z\frac{dV}{dz} &= \frac{1}{2\omega_o j}\frac{1}{1-e^{j\omega_o}z^{-1}} - \frac{1}{2\omega_o j}\frac{1}{1-e^{-j\omega_o}z^{-1}}\\
\frac{dV}{dz} &= \frac{1}{2\omega_o j}\frac{1}{z-e^{-j\omega_o}} - \frac{1}{2\omega_o j}\frac{1}{z-e^{j\omega_o}}\\
V(z) &= \frac{1}{2\omega_o j}\log\left(z-e^{-j\omega_o}\right) - \frac{1}{2\omega_o j}\log\left(z-e^{j\omega_o}\right) = \frac{1}{2\omega_o j}\log\left(\frac{z-e^{-j\omega_o}}{z-e^{j\omega_o}}\right).
\end{aligned}
$$

In summary,

$$
X(z) = 1 + \frac{1}{2\omega_o j}\log\left(\frac{1-e^{-j\omega_o}z^{-1}}{1-e^{j\omega_o}z^{-1}}\right).
$$

Problem M.5: (10 points)

Applying the definition,

$$
\begin{aligned}
(x*y)[n] &= \sum_{k\in\mathbb{Z}}\cos(\omega_o k)a^{n-k}u[n-k] = a^n\sum_{k=-\infty}^{n}\cos(\omega_o k)a^{-k}\\
&= \frac{a^n}{2}\left[\sum_{k=-\infty}^{n}(e^{j\omega_o})^k a^{-k} + \sum_{k=-\infty}^{n}(e^{-j\omega_o})^k a^{-k}\right]\\
&= \frac{a^n}{2}\left[\sum_{k=-n}^{\infty}(e^{-j\omega_o})^k a^{k} + \sum_{k=-n}^{\infty}(e^{j\omega_o})^k a^{k}\right]\\
&= \frac{a^n}{2}\left[\frac{e^{j\omega_o n}a^{-n}}{1-e^{-j\omega_o}a} + \frac{e^{-j\omega_o n}a^{-n}}{1-e^{j\omega_o}a}\right]\\
&= \frac{1}{2}\left[\frac{e^{j\omega_o n}}{1-e^{-j\omega_o}a} + \frac{e^{-j\omega_o n}}{1-e^{j\omega_o}a}\right],
\end{aligned}
$$

where the convergence is guarantees as long as $|a| < 1$.

Problem M.6: (15 points)

Since $H(z)$ is rational and it has only one simple pole at "1" and no zeros, its expression is of the form:

$$
H(z) = C\,\frac{1}{z-1},
$$

where C is a constant to be determined. Note that since the system is causal, the ROC is $|z| > 1$.

We also know that if its input is $x[n] = 2^n$ for all n, then $y[n] = 2^{n+1}$ for all n. Since a^n are eigenfunctions then we know that $y[n] = H(2)2^n$ (since $z = 2$ is in the ROC). By identification $H(2) = C = 2$. In conclusion,

$$
H(z) = \frac{2}{z-1}, \qquad |z| > 1.
$$

(a) The impulse response is

$$h[n] = 2u[n-1].$$

(b) To stabilize the system, the pole at "1" on the UC has to be canceled. The overall system function of a cascade is the product of the individual ones and therefore, it is possible to achieve stability by using a system with function

$$(z-1).$$

(c) This solution is "implementable". Indeed, it is a stable system but also even though it is not causal, it is finite length and with enough delay it becomes causal.

(d) When a system is added in parallel, the overall response is the sum

$$\frac{2}{z-1} + \frac{P(z)}{Q(z)} = \frac{2Q(z) + (z-1)P(z)}{(z-1)Q(z)}$$

and for the pole at "1" to cancel, it needs to be also a root of $Q(z)$, which means even after simplification there will remain a pole at "1".

In summary, it is not possible to obtain an overall stable system.

(e) Not applicable since the answer is "no".